Variational inequalities in Hilbert spaces with measures and optimal stopping problems

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Abstract

We study the existence theory for parabolic variational inequalities in weighted L^2 spaces with respect to excessive measures associated with a transition semigroup. We characterize the value function of optimal stopping problems for finite and infinite dimensional diffusions as a generalized solution of such a variational inequality. The weighted L^2 setting allows us to cover some singular cases, such as optimal stopping for stochastic equations with degenerate diffusion coefficient. As an application of the theory, we consider the pricing of American-style contingent claims. Among others, we treat the cases of assets with stochastic volatility and with path-dependent payoffs.

Keywords: Variational inequalities, excessive measures, Kolmogorov operators, m-accretive operators.

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1 Introduction

The aim of this work is to study a general class of parabolic variational inequalities in Hilbert spaces with suitably chosen reference measures. In particular, our motivation comes from the connection between American option pricing in mathematical finance and variational inequalities. It is well known by the classical works of Bensoussan [5] and Karatzas [18] that the price of an American contingent claim is the solution of an optimal stopping problem, whose value function can be determined, in many cases, solving an associated variational inequality (see e.g. [15] for the classical theory and [17] for connections with American options).

In this paper we study variational inequalities associated to finite and infinite dimensional diffusion processes in L^2 spaces with respect to suitably chosen measures. In particular, denoting by L the Kolmogorov operator associated to a diffusion X on a Hilbert

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space H, we shall choose a probability measure μ that is (infinitesimally) excessive for L, i.e. that satisfies $L^*\mu \leq \omega\mu$ for some $\omega \in \mathbb{R}$ (see below for precise statements). An appropriate choice of reference measure is essential in the infinite dimensional case, as there is no analog of the Lebesgue measure, and turns out to be useful also in the finite dimensional case to overcome certain limitations of the classical theory. In particular, we can relax the usual nondegeneracy assumptions on the diffusion coefficient (or on the volatility, using the language of mathematical finance), which is usually assumed in the "traditional" approach of studying variational inequalities in Sobolev spaces w.r.t. Lebesgue measure (see [6], [17]). This allows us, for instance, to characterize the price of American contingent claims on assets with degenerate or stochastic volatility as the solution of a variational inequality. Similarly, we can treat path-dependent derivatives, as well as claims on assets with certain non-Markovian price evolutions, using the infinite dimensional theory. We would like to mention that Zabczyk [25] already considered variational inequalities (called there Bellman inclusions) in weighted spaces with respect to excessive measures, including specific formulas for excessive measures and applications to American option pricing. However, some of our results on existence of solutions for the associated variatonal inequalities are more general (our assumptions on the payoff function are weaker, we allow time-dependent payoffs), and we explicitly construct a reference excessive measure in many cases of interest. Let us also recall that a study of diffusion operators in L^p spaces with respect to invariant measures (i.e. measures μ such that $L^*\mu = 0$) has been initiated in [23].

The main tool we rely on to study the above mentioned optimal stopping problems is the general theory of maximal monotone operators in Hilbert spaces. However, we need some extensions of the classical results, which are developed below and seem to be new. In particular, we establish abstract existence results for variational inequalities associated to the Kolmogorov operator of finite and infinite dimensional diffusions (on these lines see also [4] and [21]).

Variational inequalities connected to optimal stopping problems in finance have also been studied in the framework of viscosity solutions, see e.g. [22], [14]. In particular in the latter paper the authors consider the problem of optimal stopping in Hilbert space and as an application they price American interest rate contingent claims in the Gołdys-Musiela-Sondermann model. Using the approach of maximal monotone operators, at the expense of imposing only very mild additional assumptions on the payoff functions, we are able to obtain more regular solutions, which also have the attractive feature of being the limit of iterative schemes that can be implemented numerically. Moreover, the additional conditions on the payoff function we need are satisfied in essentially all situations of interest in option pricing.

The paper is organized as follows: in section 2 we prove two general existence results for the obstacle problem in Hilbert spaces. In section 3 we relate these results with the optimal stopping problem in Hilbert space. Applications to the pricing of American contingent claims are given in section 4.

2

2 Abstract existence results

Let us first introduce some notation and definitions. Given any Hilbert space E, we shall always denote by $|\cdot|_E$ its norm and by $\langle\cdot,\cdot\rangle_E$ its scalar product. Moreover, we define $C([0,T],E)$ as the space of E-valued continuous functions on $[0,T]$, and $W^{1,p}([0,T],E)$, $1 \le p \le \infty$, as the space of absolutely continuous functions $\varphi : [0,T] \to E$ with $\frac{d\varphi}{dt} \in L^p([0,T],E)$. The space of Schwarz' distributions on a domain $\Xi \subset \mathbb{R}^n$ will be denote by $\mathcal{D}'(\Xi)$. Similarly, $W^{s,p}(\Xi)$ stands for the set of functions $\phi : \Xi \to \mathbb{R}$ that are in $L^p(\Xi)$ together with their (distributional) derivatives of order up to s. Finally, $\phi \in W^{s,p}_{loc}(\Xi)$ if $\phi\zeta \in W^{s,p}$ for all $\zeta \in C^\infty_c(\Xi)$, the space of infinitely differentiable functions on Ξ with compact support.

Let H be a Hilbert space and μ be a probability measure on H. Denote by $\mathcal{H}$ the Hilbert space $L^2(H,\mu)$. Let $(P_t)_{t\ge0}$ be a strongly continuous semigroup on $\mathcal{H}$ with infinitesimal generator $-N$. We shall assume that

$$|P_t\phi|_\mathcal{H} \le e^{\omega t}|\phi|_\mathcal{H} \qquad \forall t \ge 0, \ \phi \in \mathcal{H},$$

where $\omega \in \mathbb{R}$. Then N is ω-m-accretive in $\mathcal{H}$, i.e.

$$\langle N\phi,\phi\rangle_\mathcal{H} \ge -\omega|\phi|^2_\mathcal{H} \qquad \forall\varphi \in D(N)$$

and $R(\lambda I + N) = \mathcal{H}$ for all $\lambda > \omega$, where $D(\cdot)$ and $R(\cdot)$ denote domain and range, respectively. Let $g \in \mathcal{H}$ be a given function and define the closed convex subset of $\mathcal{H}$

$$\mathcal{K}_g = \{\phi \in \mathcal{H} : \ \phi \ge g \ \mu\text{-a.e.}\}.$$

The normal cone to $\mathcal{K}_g$ at ϕ is defined by

$$\mathcal{N}_g(\phi) = \left\{z \in \mathcal{H} : \ \int_H z(\phi-\psi)\,d\mu \ge 0 \ \ \forall\psi \in \mathcal{K}\right\},$$

or equivalently

$$\mathcal{N}_g(\phi) = \left\{z \in \mathcal{H} : \ z(x) = 0 \text{ if } \phi(x) > g(x), \ z(x) \le 0 \text{ if } \phi(x) = g(x), \ \mu\text{-a.e.}\right\}.$$

We are going to study the parabolic variational inequality

$$\begin{cases} \dfrac{d\varphi}{dt}(t) + N\varphi(t) + \mathcal{N}_g(\varphi(t)) \ni f(t), & t \in (0,T) \\ \varphi(0) = \varphi_0, \end{cases} \qquad (2.1)$$

where $\varphi_0 \in \mathcal{H}$ and $f \in L^2([0,T],\mathcal{H})$ are given.

By a strong solution of (2.1) we mean an absolutely continuous function $\varphi : [0,T] \to \mathcal{H}$ which satisfies (2.1) a.e. on $(0,T)$. A function $\varphi \in C([0,T],\mathcal{H})$ is said to be a generalized

solution of (2.1) if there exist sequences $\{\varphi_0^n\} \subset \mathcal{H}$, $\{f_n\} \subset L^2([0,T],\mathcal{H})$ and $\{\varphi_n\} \subset C([0,T],\mathcal{H})$ such that, for all n, φ_n is a strong solution of

$$\frac{d\varphi}{dt}(t) + N\varphi(t) + \mathcal{N}_g(\varphi(t)) \ni f_n(t)$$

a.e. on $(0,T)$ with initial condition $\varphi(0) = \varphi_0^n$, and $\varphi_n \to \varphi$ in $C([0,T],\mathcal{H})$ as $n \to \infty$.

In order to establish existence of a solution for equation (2.1) we are going to apply the general theory of existence for Cauchy problems in Hilbert spaces associated with nonlinear maximal monotone operators (see e.g. [2], [3], [10]). We recall that the nonlinear (multivalued) operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is said to be maximal monotone (or equivalently m-accretive) if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for all $y_i \in Ax_i$, $i = 1, 2$, and $R(I + A) = \mathcal{H}$. The operator A is said to be ω-m-accretive if $\lambda I + A$ is m-accretive for all $\lambda > \omega$. If A is ω-m-accretive we set (Yosida approximation)

$$A_\lambda u = \frac{1}{\lambda}(u - (I + \lambda A)^{-1}u), \qquad u \in \mathcal{H}, \ 0 < \lambda < \frac{1}{\omega}.$$

Recall that A_λ is Lipschitz and $\frac{\omega}{1-\lambda\omega}$-accretive on $\mathcal{H}$, i.e.

$$\langle A_\lambda u - A_\lambda v, u - v \rangle_\mathcal{H} \geq -\frac{\omega}{1 - \lambda\omega}|u - v|_\mathcal{H}^2.$$

Moreover, recalling that N is ω-m-accretive, we have the following result.

Theorem 2.1. *Assume that P_t is positivity preserving (that is $P_t\varphi \geq 0$ for all $\varphi \geq 0$ μ-a.e.) and*

$$|(N_\lambda g)^+|_\mathcal{H} \leq C \qquad \forall \lambda \in (0, 1/\omega). \tag{2.2}$$

Then the operator $N + \mathcal{N}_g$ with the domain $D(N) \cap \mathcal{K}_g$ is ω-m-accretive in $\mathcal{H}$.

Proof. It is easily seen that $N + \mathcal{N}_g + \omega I$ is accretive. In order to prove m-accretivity, let us fix $f \in \mathcal{H}$ and consider the equation

$$\alpha\varphi_\lambda + N_\lambda\varphi_\lambda + \mathcal{N}_g(\varphi_\lambda) \ni f, \tag{2.3}$$

which admits a unique solution for $\alpha > \omega/(1 - \lambda\omega)$, because the operator $N_\lambda + \mathcal{N}_g + \alpha I$ is m-accretive for $\alpha > \omega/(1 - \lambda\omega)$. We are going to show that, as $\lambda \to 0$, $\varphi_\lambda \to \varphi$ strongly in $\mathcal{H}$ to a solution φ of

$$\alpha\varphi + N\varphi + \mathcal{N}_g(\varphi) \ni f. \tag{2.4}$$

Let us rewrite (2.3) as

$$\alpha\psi_\lambda + N_\lambda\psi_\lambda + \mathcal{N}_\mathcal{K}(\psi_\lambda) \ni f - \alpha g - N_\lambda g, \tag{2.5}$$

where $\psi_\lambda = \varphi_\lambda - g$, $\mathcal{K} = \{\psi \in \mathcal{H} : \ \psi \geq 0 \ \mu\text{-a.e.}\}$, and $\mathcal{N}_\mathcal{K}$ is the normal cone to $\mathcal{K}$. Setting $\eta_\lambda \in \mathcal{N}_\mathcal{K}(\psi_\lambda)$ and multiplying both sides of (2.5) by η_λ we have

$$\alpha\langle\psi_\lambda, \eta_\lambda\rangle_\mathcal{H} + |\eta_\lambda|_\mathcal{H}^2 + \langle N_\lambda\psi_\lambda + N_\lambda g, \eta_\lambda\rangle_\mathcal{H} = \langle f - \alpha g, \eta_\lambda\rangle_\mathcal{H}. \tag{2.6}$$

Since $\langle \psi_\lambda, \eta_\lambda \rangle_{\mathcal{H}} \geq 0$ (by definition of $\mathcal{N}_{\mathcal{K}}$) and $\langle N_\lambda \psi_\lambda, \eta_\lambda \rangle_{\mathcal{H}} \geq 0$ (in fact $(I + \lambda N)^{-1} \mathcal{K} \subset \mathcal{K}$ because P_t is positivity preserving), (2.6) yields

$$|\eta_\lambda|^2_{\mathcal{H}} + \langle N_\lambda g, \eta_\lambda \rangle_{\mathcal{H}} \leq \langle f - \alpha g, \eta_\lambda \rangle_{\mathcal{H}}. \tag{2.7}$$

On the other hand, we have $\langle N_\lambda g, \eta_\lambda \rangle_{\mathcal{H}} \geq \langle (N_\lambda g)^+, \eta_\lambda \rangle_{\mathcal{H}}$, because $\eta_\lambda \in \mathcal{N}_{\mathcal{K}}(\psi_\lambda)$ implies that $\langle \eta_\lambda, \phi \rangle_{\mathcal{H}} \leq 0$ if $\phi \geq 0$ μ-a.e.. Then by (2.7) and assumption (2.2) we obtain

$$|\eta_\lambda|_{\mathcal{H}} \leq |f - \alpha g|_{\mathcal{H}} + |(N_\lambda g)^+|_{\mathcal{H}} \leq C \qquad \forall \lambda \in (0, \omega^{-1}).$$

Moreover, (2.5) implies that

$$|\psi_\lambda|_{\mathcal{H}} \leq |f - \alpha g|_{\mathcal{H}} \qquad \forall \lambda \in (0, \omega^{-1}).$$

Therefore $\{\varphi_\lambda = \psi_\lambda + g\}$ and $\{\eta_\lambda\}$ are bounded in $\mathcal{H}$, and so is $\{N_\lambda \varphi_\lambda\}$. This implies by standard arguments that $\{\varphi_\lambda\}$ is Cauchy in $\mathcal{H}$, so we have that on a subsequence, again denoted by λ,

$$\begin{aligned} \varphi_\lambda &\to \varphi &&\text{strongly in } \mathcal{H}, \\ N_\lambda(\varphi_\lambda) &\to \xi &&\text{weakly in } \mathcal{H}, \\ \eta_\lambda &\to \eta &&\text{weakly in } \mathcal{H}, \end{aligned}$$

as $\lambda \to 0$. Since $\eta_\lambda \in \mathcal{N}_g(\varphi_\lambda)$ and $\mathcal{N}_g$ is maximal monotone, we have $\eta \in \mathcal{N}_g(\varphi)$ and, similarly, $\xi = N\varphi$. Hence φ is a solution of (2.4), as required. $\qquad\square$

Remark 2.2. If P_t is the transition semigroup associated to a Markov stochastic process X, then P_t is automatically positivity preserving. Assumption (2.2) holds in particular if $g \in D(N)$ or $(I + \lambda N)^{-1} g \geq g$ for all $\lambda \in (0, 1/\omega)$.

Remark 2.3. Denoting by N^* the dual of N, the operator N has a natural extension from $\mathcal{H}$ to $(D(N^*))'$ defined by $Nu(\varphi) = u(N^*\varphi)$ for all $\varphi \in D(N^*)$ and $u \in \mathcal{H}$. Then as $\lambda \to 0$ one has $N_\lambda g \to Ng$ weakly in $(D(N^*))'$ and if it happens that Ng belongs to a lattice subspace, then condition (2.2) simply means that $(Ng)^+ \in \mathcal{H}$. This is the case in spaces $L^2(\Xi)$, $\Xi \subset \mathbb{R}^n$, where usually Ng is a measure on Ξ (see e.g. [9]).

Remark 2.4. Theorem 2.1 remains true if we replace assumption (2.2) by

$$\frac{1}{t} |(g - P_t g)^+|_{\mathcal{H}} \leq C \qquad \forall t \in (0, 1). \tag{2.8}$$

The proof follows along completely similar lines.

By the general theory of Cauchy problems associated with nonlinear m-accretive operators (see e.g. [2], [3], [10]) we obtain the following result.

Theorem 2.5. *Assume that the hypotheses of Theorem 2.1 are satisfied. Let $\varphi_0 \in D(N) \cap \mathcal{K}_g$ and $f \in W^{1,1}([0,T]; \mathcal{H})$. Then there exists a unique strong solution $\varphi \in W^{1,\infty}([0,T]; \mathcal{H}) \cap L^\infty([0,T]; D(N))$ of the Cauchy problem (2.1). Moreover the function $t \mapsto \varphi(t)$ is right-differentiable and*

$$\frac{d^+}{dt} \varphi(t) + \phi(t) = 0, \qquad t \in [0,T),$$

where

$$\phi(t) = \begin{cases} N\varphi(t) - f(t) & \mu-\text{a.e. in } \{\varphi(t,x) > g(x)\} \\ (N\varphi(t) - f(t))^+ & \mu-\text{a.e. in } \{\varphi(t,x) = g(x)\}. \end{cases}$$

If $\varphi_0 \in \mathcal{K}_g$ and $f \in L^2([0,T]; \mathcal{H})$ then equation (2.1) has a unique generalized solution $\varphi \in C([0,T], \mathcal{H})$, $\varphi(t) \in \mathcal{K}_g$ for almost all $t \in [0,T]$.

We shall see later (see Theorem 2.8 below) that the generalized solution satisfies (2.1) in a more precise sense.

Remark 2.6. By the general theory of Cauchy problems for nonlinear accretive operators (see [2], [3], [10]) one knows that the solution $\varphi(t)$ given by Theorem 2.5 can be approximated as $h \to 0$ by the solution $\{\varphi_i\}_{i=1}^{N_h}$ of the finite difference scheme

$$\varphi_{i+1} + hN\varphi_{i+1} + h\mathcal{N}_g(\varphi_{i+1}) \ni f_i + \varphi_i, \qquad i = 0, 1, \dots, N_h,$$

where $hN_h = T$ and $f_i = \int_{ih}^{(i+1)h} f(t)\, dt$. Equivalently,

$$\begin{cases} \varphi_{i+1} = (I + hN)^{-1}(f_i + \varphi_i), & \text{if } (I + hN)^{-1}(f_i + \varphi_i) > g, \\ \varphi_i > g & \forall i. \end{cases}$$

2.1 Time-dependent obstacle

We shall consider the case where the obstacle function g depends also on time. In particular, we shall assume that

$$g \in W^{1,\infty}([0,T], \mathcal{H}) \tag{2.9}$$

$$\int_0^T |(N_\lambda g)^+|_{\mathcal{H}}^2\, dt \le C \qquad \forall \lambda \in (0, \omega^{-1}). \tag{2.10}$$

Let $g_\lambda = (I + \lambda N)^{-1} g$ and consider the approximating equation

$$\frac{d\varphi_\lambda}{dt}(t) + N(\varphi_\lambda(t) + g_\lambda(t) - g(t)) + \mathcal{N}_{g(t)}(\varphi_\lambda(t)) \ni f(t) \tag{2.11}$$

on $(0,T)$ with initial condition $\varphi(0) = \varphi_0$, and $\varphi_0 \ge g(0)$, $f \in L^2([0,T], \mathcal{H})$. Equivalently, setting $\psi_\lambda = \varphi_\lambda - g$, we get

$$\begin{cases} \dfrac{d\psi_\lambda}{dt}(t) + N\psi_\lambda(t) + \mathcal{N}_\mathcal{K}(\psi_\lambda(t)) \ni f(t) - \dfrac{dg}{dt}(t) - N_\lambda g(t), \\ \psi_\lambda(0) = \varphi_0 - g(0) \in \mathcal{K}. \end{cases} \tag{2.12}$$

In order to work with strong solutions of equation (2.12), we shall assume, without any loss of generality, that $f \in W^{1,1}([0,T], \mathcal{H})$, $\frac{dg}{dt} \in W^{1,1}([0,T], \mathcal{H})$, and $\varphi_0 - g(0) \in \mathcal{K} \cap D(N)$. This can be achieved in the argument which follows by taking smooth approximations of f, g and φ_0. Then equation (2.12) has a unique strong solution $\psi_\lambda \in W^{1,\infty}([0,T], \mathcal{H}) \cap L^\infty([0,T], D(N))$ by standard existence results for Cauchy problems because, as seen earlier, $N + \mathcal{N}_\mathcal{K}$ is ω-m-accretive. Moreover, multiplying both sides of (2.12) by $\eta_\lambda(t) \in \mathcal{N}_\mathcal{K}(\psi_\lambda(t))$ and taking into account that P_t is positivity preserving and

$$\int_0^T \langle N\psi_\lambda, \eta_\lambda \rangle_\mathcal{H}\, dt \geq 0, \qquad \int_0^T \left\langle \frac{d\psi_\lambda}{dt}(t), \eta_\lambda(t) \right\rangle_\mathcal{H} dt = 0 \qquad \forall \lambda \in (0, \omega^{-1}),$$

arguing as in the proof of Theorem 2.1, we get the following a priori estimates:

$$|\varphi_\lambda(t)|_\mathcal{H} \ \leq \ C \qquad \forall t \in [0,T], \tag{2.13}$$

$$\int_0^T |\eta_\lambda(t)|_\mathcal{H}^2\, dt \ \leq \ C, \tag{2.14}$$

for all $\lambda \in (0, \omega^{-1})$. Hence on a subsequence, again denoted by λ, we have

$$\begin{aligned} \varphi_\lambda &\ \rightarrow\ \varphi \quad \text{weakly* in } L^\infty([0,T], \mathcal{H}) \\ \eta_\lambda &\ \rightarrow\ \eta \quad \text{weakly in } L^2([0,T], \mathcal{H}) \end{aligned}$$

as $\lambda \to 0$. Moreover, $\varphi : [0,T] \to \mathcal{H}$ is weakly continuous and

$$\frac{d\varphi}{dt}(t) + N\varphi(t) + \eta(t) = f(t) \tag{2.15}$$

almost everywhere in $[0,T]$ with initial condition $\varphi(0) = \varphi_0$ in mild sense, i.e.,

$$\varphi(t) + \int_0^t e^{-N(t-s)} \eta(s)\, ds = e^{-Nt}\varphi_0 + \int_0^t e^{-N(t-s)} f(s)\, ds$$

for almost all $t \in [0,T]$. The latter follows by letting $\lambda \to 0$ into the equation

$$\begin{aligned} \varphi_\lambda(t) + g_\lambda(t) - g(t) + \int_0^t e^{-N(t-s)} (\eta_\lambda(s) - f(s) - g_\lambda'(s) + g'(s))\, ds \\ = e^{-Nt}(\varphi_0 + g_\lambda(0) - g(0)). \end{aligned} \tag{2.16}$$

Taking into account that, as $\lambda \to 0$, $g_\lambda(t) \to g(t)$ strongly in $\mathcal{H}$ on $[0,T]$ and $g_\lambda' - g' = (I + \lambda N)^{-1} g' - g' \to 0$ strongly in $L^2([0,T], \mathcal{H})$, we obtain the desired equation. In particular it follows that $\varphi_\lambda(t) \to \varphi(t)$ weakly in $\mathcal{H}$ for $t \in [0,T]$. We are going to show that $\eta(t) \in \mathcal{N}_g(\varphi(t))$ a.e. on $[0,T]$. To this purpose it suffices to show that

$$\limsup_{\lambda \to 0} \int_0^T e^{\gamma t} \langle \eta_\lambda(t), \varphi_\lambda(t) \rangle_\mathcal{H}\, dt \leq \int_0^T e^{\gamma t} \langle \eta(t), \varphi(t) \rangle_\mathcal{H}\, dt \tag{2.17}$$

7

for some real number γ. We shall prove that (2.17) holds with $\gamma = -2\omega$. To this end we set $N_\omega = N + \omega I$ (note that N_ω is m-accretive in $\mathcal{H}$) and, rewriting equation (2.11) as

$$\frac{d}{dt}(\varphi_\lambda + g_\lambda - g) + N_\omega(\varphi_\lambda + g_\lambda - g) + \eta_\lambda - \omega(\varphi_\lambda + g_\lambda - g) = f + g'_\lambda - g',$$

we may equivalently write (2.16) as

$$e^{-\omega t}(\varphi_\lambda(t) + g_\lambda(t) - g(t)) + \int_0^t e^{-N_\omega(t-s)}e^{-\omega s}(\eta_\lambda(s) - f(s) - g'_\lambda(s) + g'(s))\,ds$$
$$= e^{-N_\omega t}(\varphi_0 + g_\lambda(0) - g(0)) \qquad \forall t \in (0, T).$$

This yields

$$\int_0^T e^{-2\omega t}\,\langle \eta_\lambda(t), \varphi_\lambda(t)\rangle_{\mathcal{H}}\,dt =$$

$$-\int_0^T \left\langle e^{-\omega t}\eta_\lambda(t), \int_0^t e^{-N_\omega(t-s)}e^{-\omega s}\eta_\lambda(s)\right\rangle_{\mathcal{H}}ds$$

$$+\int_0^T e^{-\omega t}\,\langle \eta_\lambda(t), e^{-N_\omega t}(\varphi_0 + g_\lambda(0) - g(0)) - e^{-\omega t}(g_\lambda(t) - g(t))\rangle_{\mathcal{H}}\,dt$$

$$+\int_0^T e^{-\omega t}\left\langle \eta_\lambda(t), \int_0^t e^{-N_\omega(t-s)}e^{-\omega s}(f(s) + g'_\lambda(s) - g'(s))\,ds\right\rangle_{\mathcal{H}}\,dt$$

Then letting $\lambda \to 0$ we obtain

$$\limsup_{\lambda \to 0}\int_0^T e^{-2\omega t}\,\langle \eta_\lambda(t), \varphi_\lambda(t)\rangle_{\mathcal{H}}\,dt \leq$$

$$-\liminf_{\lambda \to 0}\int_0^T \left\langle e^{-\omega t}\eta_\lambda(t), \int_0^t e^{-N_\omega(t-s)}e^{-\omega s}\eta_\lambda(s)\,ds\right\rangle_{\mathcal{H}}\,dt \qquad (2.18)$$

$$+\int_0^T e^{-\omega t}\left\langle \eta(t), e^{-N_\omega t}\varphi_0 + \int_0^t e^{-N_\omega(t-s)}e^{-\omega s}f(s)\,ds\right\rangle_{\mathcal{H}}\,dt.$$

Consider the function

$$F(y) = \int_0^T \left\langle y(t), \int_0^t e^{-N_\omega(t-s)}y(s)\,ds\right\rangle_{\mathcal{H}}\,dt, \qquad y \in L^2([0, T], \mathcal{H}),$$

which is continuous and convex on $L^2([0, T], \mathcal{H})$ (the latter is an easy consequence of the fact that N_ω is accretive). Hence F is weakly lower semicontinuous and therefore

$$\liminf_{\lambda \to 0} F(e^{-\omega t}\eta_\lambda) \geq F(e^{-\omega t}\eta).$$

8

Substituting this expression into (2.18) we find that

$$\limsup_{\lambda \to 0} \int_0^T e^{-2\omega t} \langle \eta_\lambda(t), \varphi_\lambda(t) \rangle_{\mathcal{H}} \, dt \le$$
$$- \int_0^T \left\langle e^{-\omega t}\eta(t), \int_0^t e^{-N_\omega(t-s)} e^{-\omega s}\eta(s) \, ds \right\rangle_{\mathcal{H}} dt$$
$$+ \int_0^T e^{-\omega t} \left\langle \eta(t), e^{-N_\omega t}\varphi_0 + \int_0^t e^{-N_\omega(t-s)} e^{-\omega s} f(s) \, ds \right\rangle_{\mathcal{H}} dt \tag{2.19}$$
$$= \int_0^T e^{-2\omega t} \langle \eta(t), \varphi(t) \rangle_{\mathcal{H}} \, dt.$$

The latter follows by equation $d\varphi/dt + N\varphi + \eta = f$, or equivalently

$$\frac{d}{dt}(e^{-\omega t}\varphi(t)) + N_\omega(e^{-\omega t}\varphi(t)) + \eta(t)e^{-\omega t} = e^{-\omega t}f(t).$$

Hence $\eta(t) \in \mathcal{N}_g(\varphi(t))$ for all $t \in (0, T)$ as claimed.

Definition 2.7. *A function $\varphi \in C([0,T], \mathcal{H})$ is said to be a mild solution of*

$$\frac{d\varphi}{dt}(t) + N\varphi(t) + \mathcal{N}_{g(t)}(\varphi(t)) \ni f(t) \tag{2.20}$$

on $[0,T]$ with initial condition $\varphi(0) = \varphi_0$ if $\varphi(t) \ge g(t)$ μ-a.e. for almost all $t \in [0,T]$ and there exists $\eta \in L^2([0,T], \mathcal{H})$ with $\eta(t) \in \mathcal{N}_{g(t)}(\varphi(t))$ for almost all $t \in [0,T]$, such that

$$\varphi(t) + \int_0^t e^{-N(t-s)}\eta(s) \, ds = e^{-Nt}\varphi_0 + \int_0^t e^{-N(t-s)} f(s) \, ds \tag{2.21}$$

for all $t \in [0,T]$.

Theorem 2.8. *Assume that P_t is positivity preserving and (2.9), (2.10) hold. Let $\varphi_0 \in \mathcal{H}$, $\varphi_0 \ge g(0)$ and $f \in L^2([0,T], \mathcal{H})$. Then (2.20) has a unique mild solution. Moreover, the map $(\varphi_0, f) \mapsto \varphi$ is Lipschitz from $\mathcal{H} \times L^2([0,T], \mathcal{H})$ to $C([0,T], \mathcal{H})$.*

Proof. Existence was proved above. Uniqueness as well as as continuous dependence on data follows by (2.21) taking into account that $\eta(t) \in \mathcal{N}_{g(t)}(\varphi(t))$ for almost all $t \in [0,T]$ and

$$\int_0^T \left\langle \eta(t), \int_0^t e^{-N(t-s)}\eta(s) \, ds \right\rangle_{\mathcal{H}} dt \ge -\omega \int_0^T \left| \int_0^t e^{-N(t-s)}\eta(s) \, ds \right|^2_{\mathcal{H}} dt$$
$$+ \frac{1}{2} \left| \int_0^T e^{-N(t-s)}\eta(s) \, ds \right|^2_{\mathcal{H}}. \qquad \square$$

It is worth emphasizing that in the case where g is time-dependent the "mild" solution provided by Theorem 2.8 is a generalized solution in the sense of Theorem 2.5. However, even in this case Theorem 2.8 is not directly implied by Theorem 2.5.

3 Variational inequalities and optimal stopping problems

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$, and $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space satisfying the usual conditions, on which an H-valued Wiener process (adapted to $\mathbb{F}$) with covariance operator Q is defined. Let X be the process generated by the stochastic differential equation

$$dX(s) = b(X(s))\,ds + \sigma(X(s))\,dW(s) \tag{3.1}$$

on $s \in [t, T]$ with initial condition $X(t) = x$, where $b : H \to H$ and $\sigma : H \to L(H, H)$ are such that (3.1) admits a unique solution that is strong Markov. Define the value function $v(t, x)$ of an optimal stopping problem for X as

$$v(t, x) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_{t,x} \left[e^{-\psi(t,\tau)} g(\tau, X(\tau)) + \int_t^\tau e^{-\psi(t,s)} f(s, X(s))\,ds \right], \tag{3.2}$$

where $\mathfrak{M}$ is the family of all $\mathbb{F}$-stopping times such that $\tau \in [t, T]$ $\mathbb{P}$-a.s., and

$$\psi(t, s) = \int_t^s c(X_r)\,dr \qquad \forall t \leq s \leq T,$$

where $c : H \to \mathbb{R}_+$ is a given discount function (which we also assume to be bounded, for simplicity). Exact conditions on g and f will be specified below. The function v is formally the solution of the backward variational inequality

$$\frac{\partial u}{\partial t} + L_0 u - cu - \mathcal{N}_{g(t)}(u) \ni f \tag{3.3}$$

in $(0, T) \times H$ with terminal condition $u(T, x) = g(T, x)$, where

$$L_0 \phi = \frac{1}{2} \operatorname{Tr}[(\sigma Q^{1/2})(\sigma Q^{1/2})^* D^2 \phi] + \langle b(x), D\phi \rangle, \qquad \phi \in D(L_0) = C_b^2(H). \tag{3.4}$$

More precisely, denoting by μ an excessive measure of the transition semigroup P_t generated by the process X, we have that v is the solution of the variational inequality

$$\frac{\partial u}{\partial t} + L u - cu - \mathcal{N}_{g(t)}(u) \ni f \tag{3.5}$$

in $(0, T)$ with terminal condition $u(T) = g(T)$, where L is the infinitesimal generator of P_t. In many situations of interest $L = \overline{L_0}$, the closure of L_0 in $L^2(H, \mu)$. Before giving a simple sufficient condition for this to hold, let us define precisely excessive measures.

Definition 3.1. *Let P_t be a strongly continuous semigroup on $L^2(H, \mu)$, where μ is a probability measure on H. The measure μ is called excessive for P_t if there exists $\omega > 0$ such that*

$$\int_H P_t f\,d\mu \leq e^{\omega t} \int_H f\,d\mu \qquad \forall t \geq 0$$

for all bounded Borel functions f with $f \geq 0$ μ-a.e..

10

We have then the following result.

Lemma 3.2. *Let P_t the semigroup generated by X, and let μ be an excessive measure for P_t on H. Moreover, let $b \in C^2(H) \cap L^2(H, \mu)$, $\sigma \in C^2(H, L(H, H))$, and*

$$|Db(x)|_H + |D\sigma(x)|_{L(H,H)} \leq C \tag{3.6}$$

for all $x \in H$. Then $-L_0$ is ω-accretive and L is the closure in $L^2(H, \mu)$ of L_0 defined on $D(L_0) = C_b^2(H)$.

Proof. The argument is similar to that used in [12] for similar problems, so it will be sketched only. Fix $h \in C_b^2(H)$ and consider the equation $(\lambda I - L_0)\varphi = h$, or equivalently

$$\varphi(x) = \mathbb{E}_{0,x} \int_0^\infty e^{-\lambda t} h(X(t)) \, dt, \qquad \lambda > \omega. \tag{3.7}$$

It is readily seen that $\varphi \in C_b^2(H)$ and, by Itô's formula, $(\lambda - L_0)\varphi = h$ in H. Since $-L_0$ is closable and ω-accretive, and $R(\lambda - L_0)$ is dense in $L^2(H, \mu)$, we infer that $\overline{L_0}$ coincides with L. □

Note also that since the measure μ is ω-excessive for P_t we have $\int_H Lf \, d\mu \leq \omega \int_H f \, d\mu$, which implies that L is ω-dissipative in $L^2(H, \mu)$. In the sequel, for convenience of notation, we shall set $N = -L + cI$.

We shall further assume that $g(t, x)$ is continuously differentiable with respect to t, Lipschitz in x, and

$$\sup_{t \in (0,T)} \int_H (|D_t g(t, x)|^2 + |D_x g(t, x)|^2) \, \mu(dx) < \infty, \tag{3.8}$$

$$\mathrm{Tr}[(\sigma Q^{1/2})(\sigma Q^{1/2})^* D_{xx}^2 g] \geq 0 \qquad \text{on } (0, T) \times H. \tag{3.9}$$

If H is a finite dimensional space, the inequality (3.9) must be interpreted in the sense of distributions (i.e. of measures). In the general situation treated here the exact meaning of (3.9) is the following: there exists a sequence $\{g_\varepsilon(t)\} \subset C_b^2(H)$ such that

$$\sup_{t \in (0,T)} \int_H (|D_t g_\varepsilon(t, x)|^2 + |D_x g_\varepsilon(t, x)|^2) \, \mu(dx) < C \quad \forall \varepsilon > 0,$$

$$\mathrm{Tr}[(\sigma Q^{1/2})(\sigma Q^{1/2})^* D_{xx}^2 g_\varepsilon(t, x)] \geq 0 \qquad \qquad \forall \varepsilon > 0, \ t \geq 0, \ x \in H,$$

$$g_\varepsilon(t) \to g(t) \qquad \qquad \text{in } L^2(H, \mu) \ \forall t \geq 0.$$

It turns out that under assumption (3.9) g satisfies condition (2.10). Here is the argument: for each $\lambda > 0$ we have $(N_\lambda g)^+ = \lim_{\varepsilon \to 0} (N_\lambda g_\varepsilon)^+$ in $L^2(H, \mu)$. On the other hand, $N_\lambda g_\varepsilon = N(I + \lambda N)^{-1} g_\varepsilon$ and by (3.9) we see that

$$\mathrm{Tr}\left[(\sigma Q^{1/2})(\sigma Q^{1/2})^* D_{xx}^2 [(I + \lambda N)^{-1} g_\varepsilon]\right] \geq 0 \qquad \text{on } H$$

11

because $(I + \lambda N)^{-1}$ leaves invariant the cone of nonnegative functions (by the positivity preserving property of P_t). Hence

$$\left| (N_\lambda g_\varepsilon)^+ \right|_{L^2(H,\mu)} \leq \left| \langle b, D_x(I + \lambda N)^{-1} g_\varepsilon \rangle \right|_{L^2(H,\mu)} \leq C \qquad \forall \lambda \in (0, \omega^{-1}), \ \varepsilon > 0$$

because $b \in L^2(H, \mu)$. This implies (2.10) as claimed.

Proposition 3.3. *Assume that $f \in L^2([0,T], L^2(H,\mu) \cap C([0,T], C_b(H))$ and that conditions (2.9), (3.6) and (3.9) hold. Furthermore, assume that the law of $X(s)$ is absolutely continuous with respect to μ for all $s \in [t,T]$. Then there exists a unique mild solution $u \in C([0,T]; L^2(H,\mu))$ of the variational inequality (3.5). Moreover, u coincides μ-a.e. with the value function v defined in (3.2).*

Proof. Existence and uniqueness for (3.5) follows by Proposition 2.8. In the remaining of the proof we shall limit ourselves to the case $f = 0$. This is done only for simplicity, as the reasoning is identical in the more general case $f \neq 0$. By definition of mild solution there exists $\eta \in L^2([0,T], L^2(H,\mu))$ such that $\eta(t) \in \mathcal{N}_{g(t)}(u(t))$ for all $t \in [0,T]$ and the following equation is satisfied (in mild sense) for all $s \in (0,T)$, with terminal condition $u(T) = g(T)$:

$$\frac{du}{ds}(s) - Nu(s) = \eta(s), \tag{3.10}$$

i.e.,

$$u(t,x) = -\int_t^\tau R_{s-t}\eta(s,x)\,ds + R_{\tau-t}u(\tau,x) \quad \forall t < \tau < T, \ \mu\text{-a.e. } x \in H, \tag{3.11}$$

where R_t is the transition semigroup generated by $-N$, or equivalently the following Feynman-Kac semigroup associated with the stochastic differential equation (3.1):

$$R_t\phi(x) = \mathbb{E}_{0,x}\left[e^{-\int_0^t c(X(s))\,ds}\phi(X(t)) \right], \qquad \phi \in L^2(H,\mu).$$

Let us set $H_T = [t,T] \times H$ and define the measure $\mu_T = \text{Leb} \times \mu$ on H_T, where Leb stands for one-dimensional Lebesgue measure. Recalling that $u(s,x) \geq g(s,x)$ for all $s \in [t,T]$, μ-a.e. $x \in H$, we can obtain a version of u, still denoted by u, such that $u(s,x) \geq g(s,x)$ for all $(s,x) \in H_T$. Recalling that $\eta(s,\cdot) \in L^2(H,\mu)$ a.e. $s \in [t,T]$, equation (3.11) yields

$$u(t,x) = \mathbb{E}_{t,x}\left[\int_t^\tau -e^{-\psi(t,s)}\eta(s,X(s))\,ds + e^{-\psi(t,\tau)}u(\tau,X(\tau)) \right] \tag{3.12}$$

for every stopping time $\tau \in [t,T]$, for all $t \in [0,T]$ and μ-a.e. $x \in H$. In fact, let us consider a sequence $\{\eta_\varepsilon\} \subset C^1([0,T], \mathcal{H})$ such that $\eta_\varepsilon \to \eta$ in $L^2([0,T], \mathcal{H})$. Then equation (3.10), with η_ε replacing η, admits a solution $u_\varepsilon \in C^1([0,T], \mathcal{H}) \cap C([0,T], D(N))$ such that

$u_\varepsilon \to u$ in $C([0,T], \mathcal{H})$ as $\varepsilon \to 0$. Recalling that $N = -\overline{L_0} + cI$, there exists a sequence $\{w_\varepsilon\} \subset C^1([0,T], \mathcal{H}) \cap C([0,T], C_b^2(H))$ such that

$$|u_\varepsilon(t) - w_\varepsilon(t)|_{\mathcal{H}} \leq \varepsilon$$
$$|Nu_\varepsilon(t) - (-L_0 + cI)w_\varepsilon(t)|_{\mathcal{H}} \leq \varepsilon$$
$$|\frac{du_\varepsilon}{dt}(t) - \frac{dw_\varepsilon}{dt}(t)|_{\mathcal{H}} \leq \varepsilon$$

for all $t \in [0,T]$. Therefore we have

$$\frac{dw_\varepsilon}{dt}(t) - (L_0 - cI)w_\varepsilon(t) = \tilde{\eta}_\varepsilon \qquad \forall t \in [0,T],$$

where $\tilde{\eta}_\varepsilon \to \eta$ in $L^2([0,T], \mathcal{H})$. Then we have

$$w_\varepsilon(t,x) = \mathbb{E}_{t,x}\left[\int_t^\tau -e^{-\psi(t,s)}\tilde{\eta}_\varepsilon(s, X(s))\,ds + e^{-\psi(t,\tau)}w_\varepsilon(\tau, X(\tau))\right] \qquad (3.13)$$

for all stopping times $\tau \in [t,T]$. We shall now show that (assuming, without loss of generality, $\psi \equiv 0$)

$$\mathbb{E}_{t,x}\int_t^\tau \tilde{\eta}_\varepsilon(s, X(s))\,ds \to \mathbb{E}_{t,x}\int_t^\tau \eta(s, X(s))\,ds$$

for all $t \in [0,T]$ and in $L^2(H, \mu)$ w.r.t. x. In fact, Tonelli's theorem yields, recalling that μ is excessive for P_t,

$$\int_H \mathbb{E}_{t,x}\int_t^\tau |\tilde{\eta}_\varepsilon(s, X(s)) - \eta(s, X(s))|^2\,ds\,\mu(dx)$$
$$\leq \int_0^T \int_H \mathbb{E}_{0,x}|\tilde{\eta}_\varepsilon(s, X(s)) - \eta(s, X(s))|^2\,ds\,\mu(dx)$$
$$= \int_0^T \int_H P_s|\tilde{\eta}_\varepsilon(s,x) - \eta(s,x)|^2\,\mu(dx)\,ds$$
$$\leq e^{\omega T}\int_0^T \int_H |\tilde{\eta}_\varepsilon(s,x) - \eta(s,x)|^2\,\mu(dx)\,ds \to 0$$

as $\varepsilon \to 0$, because $\tilde{\eta}_\varepsilon \to \eta$ in $L^2([0,T], L^2(H, \mu))$. An analogous argument shows that $\mathbb{E}_{t,x}w_\varepsilon(\tau, X(\tau)) \to \mathbb{E}_{t,x}u(\tau, X(\tau))$ for all $t \in [0,T]$ and in $L^2(H, \mu)$ w.r.t. x. Therefore, passing to a subsequence of ε if necessary, we have that the left-hand and right-hand side of (3.13) converge to the left-hand and right-hand side, respectively, of (3.12) for all $t \in [0,T]$ and μ-a.e. $x \in H$. Recalling that

$$\eta(s,x) \begin{cases} = 0 & \text{if } u(s,x) > g(s,x) \quad \text{for each } s \text{ and } \mu\text{-a.e. } x \in H, \\ \leq 0 & \text{if } u(s,x) = g(s,x) \quad \text{for each } s \text{ and } \mu\text{-a.e. } x \in H, \end{cases}$$

13

let us define the set
$$A = \{(s,x) \in H_T : \ \eta(s,x) > 0\},$$

for which we have $\mu_T(A) = 0$. Using this fact together with the assumption that the law of $X(s)$ is absolutely continuous w.r.t. μ for all $s \in [t,T]$, hence that $\mathbb{P}_{t,x}((s,X(s)) \in A) = 0$, we get
$$\mathbb{E}_{t,x} \int_t^\tau -e^{-\psi(t,s)} \eta(s,X(s)) \, ds \geq 0.$$

Therefore equation (3.12) implies that $u(t,x) \geq \mathbb{E}_{t,x}[e^{-\psi(t,\tau)} g(\tau, X(\tau))]$ for all stopping times $\tau \in \mathfrak{M}$, hence $u(t,x) \geq v(t,x)$, for all $t \in [0,T]$ and μ-a.e. $x \in H$. Let us now prove that there exists a stopping time $\bar{\tau} \in [t,T]$ such that $u(t,x) = \mathbb{E}_{t,x}[e^{-\psi(t,\bar{\tau})} g(\bar{\tau}, X(\bar{\tau}))]$, which will yield $u(t,x) = v(t,x)$, for all $t \in [0,T]$, μ-a.e. $x \in H$. Define the set

$$B = \{(s,x) \in H_T : g(s,x) = u(s,x)\}$$

and the random time
$$D_B = \inf\{s \geq t : (s,X(s)) \in B\} \wedge T.$$

Since B is a Borel subset of H_T and the process $(s,X(s))$ is progressive (because it is adapted and continuous), the début theorem (see T.IV.50 in [13]) implies that D_B is a stopping time. Recalling that $u(s,x) > g(s,x)$ for all $s \in [t,D_B)$, we have, reasoning as before, $\eta(s,X(s)) = 0$ a.s. for each $s \in [t,D_B)$. Thus, taking $\bar{\tau} = D_B$, (3.12) yields

$$u(t,x) = \mathbb{E}_{t,x}[e^{-\psi(t,\bar{\tau})} g(\bar{\tau}, X(\bar{\tau}))] \qquad \forall t \in [0,T], \ \mu\text{-a.e. } x \in H.$$

We have thus proved that there exists a version of u such that $u(t,x) = v(t,x)$ for all $t \in [t,T]$, μ-a.e. $x \in H$. The definition of mild solution then implies that $u(t,x) = v(t,x)$ for all $t \in [t,T]$ and μ-a.e. $x \in H$. $\quad\square$

Remark 3.4. The absolute continuity assumption in proposition 3.3 can be difficult to verify in general. However, it holds in many cases of interest. In particular, it is automatically satisfied if the semigroup P_t is irreducible and μ is invariant with respect to P_t. Moreover, in the finite dimensional case, if the excessive measure μ is absolutely continuous with respect to Lebesgue measure and the coefficients of (3.1) satisfy an hypoellipticity condition, the assumptions of the above proposition are also satisfied. We shall see in the next section that μ has full support in all examples considered. Moreover, in the finite dimensional cases, μ can be chosen absolutely continuous with respect to Lebesgue measure. Let us also remark that the continuity of the value function has been proved under very mild assumptions by Krylov [20], and by Zabczyk [24] in the infinite dimensional case.

Remark 3.5. Optimal stopping problems in Hilbert spaces and corresponding variational inequalities are studied by Gątarek and Święch [14] in the framework of viscosity solutions. Their results are applied to pricing interest-rate American options, for which the natural

dynamics is infinite dimensional (e.g. when choosing as state variable the forward curve). At the expense of assuming (3.9), that is, roughly speaking, a convexity assumption on the payoff function g, we obtained here a more regular solution. We would like to remark that g is convex in practically all examples of interest arising in option pricing, some of which are investigated in the next section.

4   Pricing of American options

Let $\mathbb{Q}$ be a risk neutral martingale measure, and assume we have n assets whose price-per-share $X(t) = (X_i(t))_{i=1,\dots,n}$ evolves according to the following Markovian stochastic differential equation:

$$dX(t) = rX(t)\,dt + \sigma(X(t))\,dW(t), \quad X(0) = x \geq 0, \quad t \in [0,T], \tag{4.1}$$

where $r \in \mathbb{R}_+$ is the risk-free interest rate, W is a $\mathbb{R}^m$-valued Wiener process, and $\sigma : \mathbb{R}^n \to L(\mathbb{R}^m, \mathbb{R}^n)$ is the volatility function. Moreover, we assume that σ is such that $X(t) \in \mathbb{R}^n_+$ for all $t \in [0,T]$. The standard assumption (see e.g. [19]) is that $\sigma_{ij}(X(t)) = X_i(t)\tilde{\sigma}_{ij}(X(t))$ for some $\tilde{\sigma} : \mathbb{R}^n \to L(\mathbb{R}^m, \mathbb{R}^n)$. We do *not* assume, however, that σ nor $\tilde{\sigma}$ satisfies a uniform nondegeneracy condition. Note that in this situation the market is incomplete, even if $m = n$, and the choice of the risk neutral measure $\mathbb{Q}$ is not unique ([19]).

It is well known that the problem of pricing an American contingent claim with payoff function $g : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the optimal stopping problem

$$v(t,x) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_{t,x}[e^{-r\tau} g(X(\tau))], \tag{4.2}$$

where $\mathfrak{M}$ is the set of all $\mathbb{F}$-adapted stopping times $\tau \in [t,T]$ and $\mathbb{E}$ stands for expectation with respect to the measure $\mathbb{Q}$. Denote by P_t the transition semigroup associated with (4.1), i.e. $P_t f(x) = \mathbb{E}_{0,x} f(X(t))$, $f \in C_b(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and let L_0 be the corresponding Kolmogorov operator. A simple calculation based on Itô's formula yields

$$L_0 f(x) = \frac{1}{2}\operatorname{Tr}[\sigma(x)\sigma^*(x)D^2 f(x)] + \langle rx, Df(x)\rangle_{\mathbb{R}^n}, \qquad f \in C_b^2(\mathbb{R}^n).$$

By classical results (see e.g. [20]), the value function $v(t,x)$ is expected to satisfy the following backward variational inequality

$$\begin{cases} \max\Big((\partial_t + L_0)v(t,x) - rv(t,x), g(x) - v(t,x)\Big) = 0, & (t,x) \in Q_T \\ v(T,x) = g(x), & x \in \mathbb{R}^n_+, \end{cases} \tag{4.3}$$

where $Q_T = [0,T] \times \mathbb{R}^n_+$.

The classical theory of variational inequalities in Sobolev spaces with respect to Lebesgue measure does not apply, however, mainly because the volatility coefficient is degenerate (see

15

[17]). Nonetheless, one might try to study (4.3) in spaces of integrable functions with respect to a suitably chosen measure. The most natural choice would be an (infinitesimally) invariant measure for L_0. However, without non-degeneracy conditions for σ and with $r > 0$, one may not expect existence of an invariant measure (see e.g. [1], [7]). Here we shall instead solve (4.3) in $L^2(\mathbb{R}^n, \mu)$, where μ is an (infinitesimally) excessive measure for L_0, which is also absolutely continuous with respect to Lebesgue measure.

The backward variational inequality (4.3) can be equivalently written as the (abstract) variational inequality in $L^2(\mathbb{R}^n, \mu)$

$$\partial_t v - Nv - \mathcal{N}_g(v) \ni 0, \quad v(T) = g, \tag{4.4}$$

where $N = -L + rI$, with L the generator of P_t (which will often turn out to be the closure of L_0), and $\mathcal{N}_g$ is the normal cone to

$$\mathcal{K}_g = \{\phi \in L^2(\mathbb{R}^n, \mu) : \ \phi \geq g \ \mu\text{-a.s}\}.$$

Lemma 4.1. *Assume that*

$$\sigma \in C^2(\mathbb{R}^n), \quad |\sigma(x)| \leq C(1 + |x|), \quad |\sigma_{x_i}| + |\sigma_{x_i x_j}| \leq C. \tag{4.5}$$

Then there exists an excessive probability measure μ of P_t of the form

$$\mu(dx) = \frac{a}{1 + |x|^{2(n+1)}} \, dx$$

with $a > 0$.

Proof. Setting $\rho(x) = \frac{1}{1 + |x|^{2(n+1)}}$, we shall check that $L_0^* \rho \leq \omega \rho$ in $\mathbb{R}^n$ for some $\omega > 0$, where L_0^* is the formal adjoint of L_0, i.e.

$$L_0^* \rho = \frac{1}{2} \operatorname{Tr}[D^2(\sigma \sigma^* \rho)] - r \operatorname{div}(x\rho).$$

Assumption (4.5) implies, after some computations, that

$$\sup_{x \in \mathbb{R}^n} \frac{L_0^* \rho}{\rho} =: \omega < \infty,$$

thus $\mu(dx) = a\rho(x) \, dx$, with $a^{-1} = \int_{\mathbb{R}^n} \rho(x) \, dx$, is a probability measure and satisfies $L_0^* \mu \leq \omega \mu$. This yields

$$\int_{\mathbb{R}^n} L_0 f \, d\mu \leq \omega \int_{\mathbb{R}^n} f \, d\mu \tag{4.6}$$

for all $f \in C_b^2(\mathbb{R}^n)$ with $f \geq 0$, and therefore

$$\int_{\mathbb{R}^n} P_t f \, d\mu \leq e^{\omega t} \int_{\mathbb{R}^n} f \, d\mu$$

for all $f \in C_b^2(\mathbb{R}^n)$, $f \geq 0$. The latter extends by continuity to all $f \in C_b(\mathbb{R}^n)$, $f \geq 0$, and by density to all bounded Borel f with $f \geq 0$ μ-a.e.. $\qquad \square$

16

The operator L_0 is ω-dissipative in $L^2(\mathbb{R}^n, \mu)$. More precisely, we have

$$\int_{\mathbb{R}^n} (L_0 f) f \, d\mu \leq -\frac{1}{2} \int_{\mathbb{R}^n} |(\sigma\sigma^*)^{1/2} Df|^2 \, d\mu + \omega \int_{\mathbb{R}^n} f^2 \, d\mu \qquad \forall f \in C_b^2(\mathbb{R}^n),$$

as follows by (4.6) and $L_0(f^2) = 2(L_0 f)f + |(\sigma\sigma^*)^{1/2} Df|^2$.
Note also that for each $h \in C_b^2(\mathbb{R}^n)$ the function

$$\varphi(x) = \mathbb{E}_{0,x} \int_0^\infty e^{-\lambda t} h(X(t)) \, dt \tag{4.7}$$

is in $C_b^2(\mathbb{R}^n)$ and satisfies the equation

$$\lambda\varphi - L_0\varphi = h$$

in $\mathbb{R}^n$. Hence $R(\lambda I - L_0)$ is dense in $L^2(\mathbb{R}^n, \mu)$ and since L_0 is closable, its closure $L := \overline{L_0}$ is ω-m-dissipative, i.e. $-\omega I + L$ is m-dissipative. Since, by (4.7), $(\lambda I - L)^{-1}$ is the resolvent of the infinitesimal generator of P_t, we also infer that L is just the infinitesimal generator of P_t. We have thus proved the following result.

Lemma 4.2. *The infinitesimal generator of P_t in $L^2(\mathbb{R}^n, \mu)$ is L. Moreover one has*

$$\int_{\mathbb{R}^n} (Lf) f \, d\mu \leq -\frac{1}{2} \int_{\mathbb{R}^n} |\sigma^* Df|^2 \, d\mu + \omega \int_{\mathbb{R}^n} f^2 \, d\mu$$

for all $f \in L^2(\mathbb{R}^n, \mu)$.

Taking into account that L is the closure (i.e. Friedrichs' extension) of L_0 in $L^2(\mathbb{R}, \mu)$, it follows that for each $f \in D(L)$ we have

$$Lf = \frac{1}{2} \operatorname{Tr}[\sigma\sigma^* D^2 f] + \langle rx, Df \rangle_{\mathbb{R}^n}$$

in $\mathcal{D}'(\mathbb{R}^n)$, where Df, $D^2 f$ are taken in the sense of distributions. In particular, it follows by the previous lemma that

$$(\sigma\sigma^*)^{1/2} f \in W^{1,2}(\mathbb{R}^n, \mu), \qquad f \in W^{2,2}_{loc}(\Xi)$$

for each $f \in D(\overline{L})$, where $\Xi = \{x \in \mathbb{R}^n : \operatorname{Tr}[\sigma\sigma^*](x) > 0\}$.

We are now going to apply Theorem 2.1 to the operator $N = -L + rI$ on the set

$$\mathcal{K}_g = \left\{ \varphi \in L^2(\mathbb{R}^n, \mu) : \varphi(x) \geq g(x) \ \mu\text{--a.e.} \right\}.$$

The function $g : \mathbb{R}^n \to \mathbb{R}$ is assumed to satisfy the following conditions:

$$Dg \in L^\infty(\mathbb{R}^n), \qquad \operatorname{Tr}[\sigma\sigma^* D^2 g] \in \mathcal{M}(\mathbb{R}^n), \qquad \operatorname{Tr}[\sigma\sigma^* D^2 g] \geq 0 \ \text{ in } \mathcal{M}(\mathbb{R}^n), \tag{4.8}$$

17

where $\mathcal{M}(\mathbb{R}^n)$ is the space of bounded Radon measures on $\mathbb{R}^n$.

Payoff functions that can be covered in this setting include so-called Margrabe options (with payoff $g(x) = (x_i - \lambda x_j)^+$, for given $\lambda > 0$ and $i \neq j \leq n$) and basket put options. We shall focus, as an illustration of the theory, on the latter case, for which

$$g(x_1, \ldots, x_n) = \Big(k - \sum_{j=1}^n \lambda_j x_j \Big)^+, \qquad \sum_{j=1}^n \lambda_j = 1.$$

In this case the first two conditions in (4.8) are obviously satisfied and

$$\mathrm{Tr}[\sigma \sigma^* D^2 g] = \Big(\sum_{i,j=1}^n a_{ij} \lambda_i \lambda_j \Big) \delta \geq 0$$

in $\mathcal{M}(\mathbb{R}^n)$, where $a = \sigma \sigma^*$ and δ is the Dirac measure. Moreover,

$$D_i g(x) = -\lambda_i H \Big(r - \sum_{j=1}^n \lambda_j x_j \Big), \qquad i = 1, \ldots, n,$$

where H is the Heaviside function, i.e. $H(r) = 1$ for $r \geq 0$ and $H(r) = 0$ otherwise. The operator N has a natural extension to functions g satisfying the first two conditions in (4.8) through the formula

$$(Ng)\varphi = \int_{\mathbb{R}^n} g N^* \varphi \, d\mu \qquad \forall \varphi \in D(N^*).$$

In our case one has

$$Ng = -\frac{1}{2} \Big(\sum_{i,j=1}^n a_{ij} \lambda_i \lambda_j \Big) \delta - r \sum_{j=1}^n \lambda_j x_j \, H \Big(k - \sum_{j=1}^n \lambda_j x_j \Big) + rg.$$

Lemma 4.3. *Assume that (4.5) and (4.8) are verified. Then the operator $N + \mathcal{N}_g$ is ω-m-accretive in $L^2(\mathbb{R}^n, \mu)$.*

Proof. We only have to prove that $|(N_\lambda g)^+|_{L^2(\mathbb{R}^n, \mu)}$ is bounded for all $\lambda \in (0, \omega^{-1})$, as required by Theorem 2.1. Set $g_\lambda = (I + \lambda N)^{-1} g$, i.e.

$$g_\lambda + \lambda N g_\lambda = g, \qquad N_\lambda g = N g_\lambda. \tag{4.9}$$

Then we have

$$(1 + \lambda r) g_\lambda(x) - \frac{\lambda}{2} \sum_{i,j=1}^n a_{ij} D_{ij}^2 g_\lambda(x) - r \sum_{i=1}^n x_i D_i g_\lambda(x) = g(x)$$

18

in $\mathcal{D}'(\mathbb{R}^n)$. As seen earlier, $Ng = -\frac{1}{2}\operatorname{Tr}[\sigma\sigma^* D^2 g] - \langle rx, Dg \rangle + rg$ in $\mathcal{D}'(\mathbb{R}^n)$ and by assumption (4.8) we have that $(Ng)^+ = (-r\langle x, Dg \rangle + rg)^+$ (where ν^+ denotes the positive part of the measure ν). Since $Dg \in L^\infty(\mathbb{R}^n, dx)$ we conclude that $(Ng)^+ \in L^2(\mathbb{R}^n, \mu)$. Approximating g by a sequence $g_\varepsilon \in D(N)$ we may assume that $g \in D(N)$ and also $Ng_\lambda \in D(N)$. We set $\psi_\lambda = Ng_\lambda$ and so (4.9) yields

$$(1+\lambda r)\psi_\lambda(x) - \frac{\lambda}{2}\sum_{i,j=1}^n a_{ij}(x)D_{ij}^2\psi_\lambda(x) - r\sum_{i=1}^n x_i D_i\psi_\lambda(x) = Ng(x).$$

Let us set $\psi_\lambda = \psi_\lambda^1 + \psi_\lambda^2$, with

$$(1+\lambda r)\psi_\lambda^1(x) - \frac{\lambda}{2}\sum_{i,j=1}^n a_{ij}D_{ij}^2\psi_\lambda^1 - r\sum_{i=1}^n x_i D_i\psi_\lambda^1 \;=\; (Ng)^+$$

$$(1+\lambda r)\psi_\lambda^2(x) - \frac{\lambda}{2}\sum_{i,j=1}^n a_{ij}D_{ij}^2\psi_\lambda^2 - r\sum_{i=1}^n x_i D_i\psi_\lambda^2 \;=\; (Ng)^-,$$

where the first equation is taken in $L^2(\mathbb{R}^n, \mu)$ and the second in $\mathcal{D}'(\mathbb{R}^n)$. By the maximum principle for elliptic equations we infer that $\psi_\lambda^1 \ge 0$, $\psi_\lambda^2 \ge 0$, hence $\psi_\lambda^1 = \psi_\lambda^+$ and $\psi_\lambda^2 = \psi_\lambda^-$. This implies that $\psi_\lambda^+ = (N_\lambda g)^+$ is the solution ψ_λ^1 of

$$\psi_\lambda^1 + \lambda N\psi_\lambda^1 = (Ng)^+.$$

But the solution of this equation satisfies

$$|\psi_\lambda^1|_{L^2(\mathbb{R}^n,\mu)}^2 \le \frac{|(Ng)^+|_{L^2(\mathbb{R}^n,\mu)}}{1-\lambda\omega},$$

hence $\{(N_\lambda g)^+\}_\lambda$ is bounded as claimed. $\qquad\square$

Applying Corollary 2.5 we obtain the following existence result for the value function of the optimal stopping problem, i.e. for the price of the American option.

Corollary 4.4. *Assume that conditions (4.5), (4.8) hold and that $g \in \overline{D(N)} = L^2(\mathbb{R}^n, \mu)$. Then the backward variational inequality associated to the optimal stopping problem (4.2), i.e.*

$$\begin{cases} \dfrac{\partial u}{\partial t} - Nu - \mathcal{N}_g(u) \ni 0, & \text{a.e. } t \in (0,T), \\ u(T) = g, \end{cases} \qquad (4.10)$$

admits a unique generalized (mild) solution u in $C([0,T]; L^2(\mathbb{R}^n, \mu))$. Moreover, if $g \in D(N)$, then $u \in W^{1,\infty}([0,T]; L^2(\mathbb{R}^n, \mu))$ is the unique strong solution of (4.10). Furthermore, if the law of the solution of (4.1) is absolutely continuous with respect to μ, then the value function v coincides with u for all $s \in [t,T]$ and μ-a.e. $x \in \mathbb{R}^n$.

19

Let us remark that the last assertion of the corollary is included for completeness only, as we do not know of any option whose payoff g is smooth enough so that $g \in D(N)$. On the other hand, the general case $g \in \overline{D(N)}$ covered in the corollary happens for virtually all payoff functions g. Then the solution is just the limit of the following backward finite difference scheme:

$$v_i = \theta_{M-i}, \quad \theta_{i+1} + hN\theta_{i+1} + \mathcal{N}_g(\theta_{i+1}) \ni \theta_i, \quad \theta_0 = v_0, \quad h = T/M.$$

This discretized elliptic variational inequality can be solved via the penalization scheme

$$\theta_{i+1}^\varepsilon + hN\theta_{i+1}^\varepsilon - \frac{1}{\varepsilon}(\theta_{i+1}^\varepsilon - g)^- = \theta_i^\varepsilon, \qquad i = 0, 1, \ldots, M-1,$$

or via the bounded penalization scheme (see e.g. [8])

$$\theta_{i+1}^\varepsilon + hN\theta_{i+1}^\varepsilon + g_1 \frac{\theta_{i+1}^\varepsilon - g}{\varepsilon + |\theta_{i+1}^\varepsilon - g|} = \theta_i^\varepsilon + g_1, \qquad i = 0, 1, \ldots, M-1,$$

where g_1 is an arbitrary parameter function. Therefore the characterization of the option price given by Corollary 4.4 is also constructive, that is, it is guaranteed to be the unique limit of very natural finite difference approximation schemes, that can be implemented numerically. A completely analogous remark applies also to the cases treated in the next subsections.

4.1   American options on assets with stochastic volatility

Consider the following model of asset price dynamics with stochastic volatility under a risk neutral measure $\mathbb{Q}$:

$$
\begin{aligned}
dX(t) &= \sqrt{V(t)}X(t)\,dW_1(t) \\
dV(t) &= \kappa(\theta - V(t))\,dt + \eta\sqrt{V(t)}\,dW_2(t),
\end{aligned}
$$

where $W(t) = (W_1(t), W_2(t))$ is a 2-dimensional Wiener process with identity covariance matrix (the more general case of correlated Wiener processes is completely analogous), κ, θ, η are positive constants, and the risk-free interest rate is assumed to be zero. Moreover, in order to ensure that $V(t) \geq 0$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$, we assume that $2\kappa\theta > \eta^2$ (see e.g. [16]).

It is convenient to use the transformation $x(t) = \log X(t)$, after which we can write (by a simple application of Itô's lemma)

$$dx(t) = -V(t)/2\,dt + \sqrt{V(t)}\,dW_1(t).$$

Define $Y(t) = (x(t), V(t))$. Then we have

$$dY(t) = A(Y(t)) + G(Y(t))\,dW(t), \qquad (4.11)$$

20

where $A : \mathbb{R}^2 \ni (x, v) \mapsto (-v/2, \kappa(\theta - v)) \in \mathbb{R}^2$ and $G : \mathbb{R}^2 \ni (x, v) \mapsto \text{diag}(\sqrt{v}, \eta\sqrt{v}) \in L(\mathbb{R}^2, \mathbb{R}^2)$. The price of an American contingent claim on X with payoff function $g : \mathbb{R} \to \mathbb{R}$ is the value function v of an optimal stopping problem, namely

$$v(t, x, v) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_{t,(x,v)}[\tilde{g}(Y(\tau))], \tag{4.12}$$

where $\tilde{g}(x, v) \equiv g(e^x)$ and $\mathfrak{M}$ is the set of all stopping times τ such that $\tau \in [s, T]$ $\mathbb{Q}$-a.s..

The Kolmogorov operator L_0 associated to (4.11) is given by

$$L_0 f = \frac{1}{2} v f_{xx} + \frac{1}{2} \eta^2 v f_{vv} - \frac{1}{2} v f_x + \kappa(\theta - v) f_v, \qquad f \in C_b^2(\mathbb{R}^2),$$

and its adjoint L_0^* takes the form

$$L_0^* \rho = \frac{1}{2} v \rho_{xx} + \frac{1}{2} \eta^2 (v\rho)_{vv} + \frac{1}{2} v \rho_x - \kappa((\theta - v)\rho)_v, \qquad \rho \in C_b^2(\mathbb{R}^2), \tag{4.13}$$

Following the same strategy as above, we look for an excessive measure of the form

$$\mu(dx, dv) = a\rho(x, v)\, dx\, dv, \qquad \rho(x, v) = \frac{1}{1 + x^2 + v^2},$$

where $a^{-1} = \int_{\mathbb{R}^2} \rho(x, v)\, dx\, dv$.

Some calculations involving (4.13) reveal that

$$\sup_{(x,v) \in \mathbb{R} \times \mathbb{R}_+} \frac{L_0^* \rho(x, v)}{\rho(x, v)} = \omega < \infty,$$

i.e. μ is an infinitesimally excessive measure for L_0 on $\Xi = \mathbb{R} \times \mathbb{R}_+$. Then the transition semigroup

$$P_t f(x, v) = \mathbb{E}_{0,(x,v)} f(x(t), V(t)), \qquad f \in C_b^2(\Xi),$$

extends by continuity to $L^2(\Xi, \mu)$, and the operator L_0 with domain $C_b^2(\Xi))$ is ω-dissipative in $L^2(\Xi, \mu)$. Arguing as above (see Lemma 3.2), the closure L of L_0 is ω-m-dissipative in $L^2(\Xi, \mu)$ and

$$\int_\Xi (Lf) f\, d\mu \le -\frac{\eta^2 + 1}{2} \int_\Xi v(f_x^2 + f_v^2)\, d\mu + \omega \int_\Xi f^2\, d\mu.$$

The operator $N = -L$ is therefore ω-m-accretive and formally one has

$$\begin{aligned} N\tilde{g} &= -\frac{1}{2} v(e^{2x} g''(e^x) + e^x g'(e^x)) + \frac{1}{2} v e^x g'(e^x) \\ &= -\frac{1}{2} v e^{2x} g''(e^x). \end{aligned} \tag{4.14}$$

The previous expression is of course rigorous if g is smooth and $N\tilde{g} \in L^2(\Xi, \mu)$, but in general (i.e. for $\tilde{g} \in L^2(\Xi, \mu)$) is has to be interpreted in the sense of distributions on Ξ in order to be meaningful.

21

We shall assume that the payoff function g is convex on $\mathbb{R}$, more precisely,

$$g'' \in \mathcal{M}(\mathbb{R}), \qquad g'' \geq 0, \tag{4.15}$$

where $\mathcal{M}(\mathbb{R})$ is the space of finite measures on $\mathbb{R}$. Note that the typical payoff of a put or call option is covered by these assumptions. Equation (4.14) implies that $N\tilde{g} \in \mathcal{D}'(\mathbb{R})$ and $N\tilde{g} \leq 0$ in $\mathcal{D}'(\mathbb{R})$, hence $N\tilde{g}$ is a negative measure and so the hypotheses of Theorem 2.1 are met. Thus, defining $\mathcal{K}_g = \{\varphi \in L^2(\Xi, \mu) : \quad \varphi \geq \tilde{g} \quad \mu\text{-a.e.}\}$, it follow that the operator $N + \mathcal{N}_g$ is ω-m-accretive on $H = L^2(\Xi, \mu)$. This yields

Corollary 4.5. *Assume that (4.15) holds. Then the backward variational inequality*

$$\frac{\partial u}{\partial t} - Nu - \mathcal{N}_g(u) \ni 0 \tag{4.16}$$

on $H_T = [0,T] \times L^2(\Xi, \mu)$ with terminal condition $u(T) = \tilde{g}$ has a unique generalized (mild) solution $u \in C([0,T], L^2(\Xi, \mu))$. Moreover, if $g \in D(N)$, then (4.16) has a unique strong solution $u \in W^{1,\infty}([0,T], L^2(\Xi, \mu))$. Furthermore, if the law of the solution of (4.11) is absolutely continuous with respect to μ, then the value function v defined in (4.12) coincides with u for all $s \in [t,T]$ and μ-a.e. $(x,v) \in \Xi$.

4.2 Asian options with American feature

Let the price process X of a given asset satisfy the following stochastic differential equation, under an equivalent martingale measure $\mathbb{Q}$:

$$dX = rX\,dt + \sigma(X)\,dW(t), \qquad X(0) = x.$$

Here we consider the problem of pricing a "regularized" Asian options with American feature, that is we look for the value function v of the optimal stopping problem

$$v(x) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_x \left(k - \frac{1}{\tau + \delta} \int_0^\tau X_s\,ds \right)^+, \tag{4.17}$$

where $k \geq 0$ is the strike price, $\delta > 0$ is a "small" regularizing term, $\mathfrak{M}$ is the set of stopping times between 0 and T, and $\mathbb{E}_x$ stands for expectation w.r.t. $\mathbb{Q}$, conditional on $X(0) = x$. The standard Asian payoff corresponds to $\delta = 0$. Unfortunately we are not able to treat with our methods this limiting situation, as it gives rise to a singularity in the obstacle function of the associated variational inequality, or, in the approach we shall follow here, in the Kolmogorov operator of an associated stochastic system. However, it is clear that for small values of δ the value function v in (4.17) is a good approximation of the option price, at least for optimal exercise times that are not of the same order of magnitude of δ.

Let us define the auxiliary processes

$$Y(t) = \frac{1}{t + \delta} \int_0^t X(s)\,ds$$

and $S(t) = t$. Then we have

$$
\begin{cases}
dX(t) = rX(t)\, dt + \sigma(X(t))\, dW(t) \\
dY(t) = \dfrac{X(t) - Y(t)}{S(t) + \delta}\, dt \\
dS(t) = dt
\end{cases}
$$

with initial conditions $X(0) = x$, $Y(0) = 0$, $s(0) = 0$. This system can be equivalently written in terms of the vector $Z = (X, Y, S)$ as

$$
dZ(t) = A(Z(t))\, dt + G(Z(t))\, dW(t), \qquad Z(0) = (x, 0, 0), \qquad (4.18)
$$

where $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A : (x, y, s) \mapsto (rx, (s + \delta)^{-1}(x - y), 1)$ and $G : \mathbb{R}^3 \to L(\mathbb{R}, \mathbb{R}^3) \simeq \mathbb{R}^3$, $G(x, y, s) = (\sigma(x), 0, 0)$. Therefore (4.17) is equivalent to

$$
v(x) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_x g(Z(\tau)),
$$

where $g : (x, y, s) \to (k - y)^+$ and $\mathbb{E}_x$ stands for $\mathbb{E}_{(x,0,0)}$.

As in the previous cases, we shall look for an excessive measure of L_0, the Kolmogorov operator associated to (4.18), which is given by

$$
L_0 f = \frac{1}{2}\sigma^2(x) D_{xx}^2 f + rx D_x f + \frac{x - y}{s + \delta} D_y f + D_s f, \qquad f \in C_b^2(\mathbb{R}^3).
$$

Then the adjoint of L_0 can be formally written as

$$
L_0^* \rho = \frac{1}{2} D_{xx}^2 (\sigma^2(x)\rho) - r D_x(x\rho) - D_y\left(\frac{x - y}{s + \delta}\rho\right) - D_s \rho.
$$

In analogy to previous cases, some calculations reveal that, under the assumptions (4.5) on σ, there exists a measure μ of the type $\mu(dx, dy, ds) = \rho(x, y, s)\, dx\, dy\, ds$,

$$
\rho(x, y, s) = \frac{a}{(1 + |x|)^{2n+1}(1 + |x - y|)^{2n+1}(1 + s)^2}, \qquad a^{-1} = \int_{\mathbb{R}^3} \rho(z)\, dz,
$$

such that $L_0^* \rho \leq \omega \rho$ for some $\omega \in \mathbb{R}$. Arguing as before, we conclude that μ is an excessive measure for the semigroup P_t generated by the stochastic equation (4.18), and that L, the closure of L_0 in $L^2(\mathbb{R}^3, \mu)$, is the infinitesimal generator of P_t.

We are now in the setting of section 3, i.e. we can characterize the option price as the (generalized) solution of a suitable variational inequality. Details are left to the reader.

4.3 Path-dependent American options

We shall consider a situation where the price dynamics is non-Markovian as it may depend on its history, and the payoff function itself is allowed to depend on past prices. We should remark, however, that in the present setup we still cannot cover Asian options of the type discussed in the previous subsection, with $\delta = 0$.

Consider the following price evolution of n assets under a risk-neutral measure $\mathbb{Q}$:

$$\begin{cases} dX(t) = rX(t) + \sigma(X(t), X_s(t))\, dW(t), & 0 \le t \le T \\ X(0) = x_0, \qquad X_s(0) = x_1(s), & -T \le s \le 0, \end{cases} \tag{4.19}$$

where $X_s(t) = X(t + s)$, $s \in (-T, 0)$, W is a standard Wiener process on $\mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times L^2([-T, 0], \mathbb{R}^n) \to L(\mathbb{R}^n, \mathbb{R}^n)$. Let us consider an American contingent claim with payoff of the type $g : \mathbb{R}^n \times L^2([-T, 0] \to \mathbb{R}$, whose price is equal to the value function of the optimal stopping problem

$$v(s, x_0, x_1) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_{s,(x_0,x_1)}[e^{-r\tau} g(X(\tau), X_s(\tau))],$$

where the notation is completely analogous to the previous subsection. An example that can be covered by this functional setting is $g(x_0, x_1) = \alpha_0 g_0(x_0) + \alpha_2 g_1(x_1)$, with $\alpha_1, \alpha_2 \ge 0$ and $g_0(x_0) = (k_0 - x_0)^+$ and $g_1(x_1) = (k_1 - \int_{-T}^0 x_1(s)\, ds)^+$.

Let us now rewrite (4.19) as an infinite dimensional stochastic differential equation on the space $H = \mathbb{R}^n \times L^2([-T, 0], \mathbb{R}^n)$. Define the operator $A : D(A) \subset H \to H$ as follows:

$$\begin{aligned} A : (x_0, x_1) &\mapsto (rx_0, x_1') \\ D(A) &= \{(x_0, x_1) \in H; \ x_1 \in W^{1,2}((-T, 0), \mathbb{R}^n), \ x_1(0) = x_0\}. \end{aligned}$$

Setting $G(x_0, x_1) = (\sigma(x_0, x_1), 0)$, let us consider the stochastic differential equation on H

$$dY(t) = AY(t)\, dt + G(Y(t))\, dW(t) \tag{4.20}$$

with initial condition $Y(0) = (x_0, x_1)$. The evolution equation (4.20) is equivalent to (4.19) in the following sense (see [11]): if X is the unique solution of (4.19), then $Y(t) = (X(t), X_s(t)$ is the solution of (4.20). Note that (4.20) has a unique solution if G is Lipschitz on H. Finally, regarding g as a real-valued function defined on H, we are led to study the optimal stopping problem in the Hilbert space H

$$v(s, x) = \sup_{\tau \in \mathfrak{M}} \mathbb{E}_{s,x}[e^{-r\tau} g(Y(\tau))].$$

The Kolmogorov operator L_0 associated to (4.20) has the form, on $C_b^2(H)$,

$$\begin{aligned} L_0 \varphi(x_0, x_1) &= \frac{1}{2} \operatorname{Tr}[\sigma\sigma^*(x_0, x_1) D_{x_0}^2 \varphi(x_0, x_1)] + \langle rx_0, D_{x_0}\varphi(x_0, x_1)\rangle_{\mathbb{R}^n} \\ &\quad + \int_{-T}^0 \langle x_1(s), D_{x_1}\varphi(x_0, x_1(s))\rangle_{\mathbb{R}^n}\, ds. \end{aligned}$$

24

We look for an excessive measure μ for L_0 of the form $\mu = \nu_1 \otimes \nu_2$, where ν_1, ν_2 are probability measures on $\mathbb{R}^n$ and $L^2([-T, 0], \mathbb{R}^n)$, respectively. In particular, we choose

$$\nu_1(dx_0) = \rho(x_0)\, dx_0, \qquad \rho(x_0) = \frac{a}{1 + |x_0|^{2n}}, \qquad a = \left(\int_{\mathbb{R}^n} \frac{1}{1 + |x_0|^{2n}}\, dx_0 \right)^{-1},$$

and ν_2 a Gaussian measure on $L^2([-T, 0], \mathbb{R}^n)$. Setting $H_0 = L^2([-T, 0], \mathbb{R}^n)$, a simple calculation reveals that

$$
\begin{aligned}
\int_H L_0 \varphi\, d\mu &= \frac{1}{2} \int_{H_0} d\nu_2 \int_{\mathbb{R}^n} \mathrm{Tr}[\sigma \sigma^* D_{x_0}^2 \varphi]\, d\nu_1 + r \int_{H_0} d\nu_2 \int_{\mathbb{R}^n} \langle x_0, D_{x_0} \varphi \rangle_{\mathbb{R}^n}\, d\nu_1 \\
&\quad + \int_{\mathbb{R}^n} d\nu_1 \int_{H_0} \langle x_1, D_{x_1} \varphi \rangle_{H_0}\, d\nu_2 \\
&= \frac{1}{2} \int_{H_0} d\nu_2 \int_{\mathbb{R}^n} \varphi D_{x_0}^2 (\sigma \sigma^* \rho)\, dx_0 - r \int_{H_0} d\nu_2 \int_{\mathbb{R}^n} \varphi D_{x_0}(x_0 \rho)\, dx_0 \\
&\quad + \int_{\mathbb{R}^n} d\nu_1 \int_{H_0} \langle x_1, D_{x_1} \varphi \rangle_{H_0}\, d\nu_2.
\end{aligned}
\tag{4.21}
$$

We shall assume that

$$
\begin{aligned}
&\sigma \in C^2(\mathbb{R}^n \times L^2([-T, 0], \mathbb{R}^n)) \cap \mathrm{Lip}(\mathbb{R}^n \times L^2([-T, 0], \mathbb{R}^n)), \\
&\sigma(x_0, x_1) \leq C(|x_0| + |x_1|_{H_0}) \qquad \forall (x_0, x_1) \in H, \\
&|\sigma_{x_i}(x_0, x_1)| + |\sigma_{x_i x_j}(x_0, x_1)| \leq C \quad \forall (x_0, x_1) \in H, \ i, j = 1, 2.
\end{aligned}
\tag{4.22}
$$

Note that these conditions also imply existence and uniqueness of a solution for (4.19). Taking into account that $\int_{H_0} |x_1|^{2m}\, d\nu_2 < \infty$ and that

$$\int_{H_0} \langle x_1, D_{x_1} \varphi \rangle_{H_0}\, d\nu_2 = -\int_{H_0} \varphi(n - \langle Q^{-1} x_1, x_1 \rangle_{H_0})\, d\nu_2$$

(where Q is the covariance operator of ν_2), we see by (4.21) and (4.22) that there exists $\omega \geq 0$ such that

$$\int_H L_0 \varphi\, d\mu \leq \omega \int_H \varphi\, d\mu \tag{4.23}$$

for all $\varphi \in C_b^2(H)$, $\varphi \geq 0$.

The operator L_0 is thus closable and ω-dissipative in $L^2(H, \mu)$. Moreover, (4.23) implies that

$$\int_H (L_0 \varphi) \varphi\, d\mu \leq -\frac{1}{2} \int_H |(\sigma \sigma^*)^{1/2} D_{x_0} \varphi|^2\, d\mu + \omega \int_H \varphi^2\, d\mu \quad \forall \varphi \in C_b^2(H). \tag{4.24}$$

Since one has, for $\lambda > \omega$,

$$(\lambda I - L_0)^{-1} \varphi = \mathbb{E} \int_0^\infty e^{-\lambda t} \varphi(X(t), X_s(t))\, dt \quad \forall \varphi \in C_b^2(H),$$

we infer that $R(\lambda I - L_0)$ is dense in $L^2(H, \mu)$ and so the closure L of L_0 is ω-m-dissipative in $L^2(H, \mu)$, and it is the infinitesimal generator of the transition semigroup P_t defined by (4.20). We set $N = -L + rI$.

Furthermore, let us assume that

$$g(\cdot, x_1) \in \mathrm{Lip}(\mathbb{R}^n), \qquad D_{x_0}^2 g(\cdot, x_1) \in \mathcal{M}(\mathbb{R}^n) \quad \forall x_1 \in H_0, \tag{4.25}$$

$$\mathrm{Tr}[\sigma\sigma^* D_{x_0}^2 g](\cdot, x_1) \geq 0 \quad \forall x_1 \in H_0, \tag{4.26}$$

where (4.26) is taken in the sense of distributions (or equivalently in the sense of $\mathcal{M}(\mathbb{R}^n)$). This implies, as in previous cases, that condition (2.10) is satisfied.

In particular, note that (4.25) and (4.26) hold if $g = \alpha_0 g_0 + \alpha_1 g_1$, as in the example mentioned above. Assumptions (4.25) and (4.26) imply that

$$
\begin{aligned}
Ng &= -\frac{1}{2}\mathrm{Tr}[\sigma\sigma^* D^2 g] - r \langle x_0, D_{x_0} g \rangle_{\mathbb{R}^n} - \langle x_1, D_{x_1} g \rangle_{H_0} + rg \\
&\leq -r \langle x_0, D_{x_0} g \rangle_{\mathbb{R}^n} - \langle x_1, D_{x_1} g \rangle_{H_0} + rg,
\end{aligned}
$$

hence $(Ng)^+ \in L^2(H, \mu)$, because $\langle x_0, D_{x_0} g \rangle_{\mathbb{R}^n}$, $\langle x_1, D_{x_1} g \rangle_{H_0}$, $g \in L^2(H, \mu)$.

Once again the results established in sections 3 allow us to characterize the price of the American option as solution (mild, in general, as the typical payoff function g is not smooth) of the backward variational inequality on $[0, T]$

$$\frac{d\varphi}{dt} - N\varphi - \mathcal{N}_g(\varphi) \ni 0$$

with terminal condition $\varphi(T) = g$, where $\mathcal{N}_g$ is the normal cone to

$$\mathcal{K}_g = \{x \in H : \ \varphi(x) \geq g(x) \ \mu\text{-a.e.}\}.$$

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